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B.Sc-part II(H), paper-III, Date-20-04-20

Differential Calculus.

Q.1. If $u = \log(x^2+y^2)$ show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Solⁿ: We have, $u = \log(x^2+y^2)$

Differentiating partially w.r.t. x, we get

$$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2} \times 2x = \frac{2x}{x^2+y^2}$$

Again differentiating partially w.r.t. x, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{2[1\cdot(x^2+y^2) - 2x\cdot x]}{(x^2+y^2)^2} = \frac{2[x^2+y^2-2x^2]}{(x^2+y^2)^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{2(y^2-x^2)}{(x^2+y^2)^2} \quad \text{——— I}$$

Similarly, diff. partially w.r.t. y, we get

$$\frac{\partial u}{\partial y} = \frac{2y}{x^2+y^2}$$

Again, differentiating partially w.r.t. y, we get

$$\frac{\partial^2 u}{\partial y^2} = \frac{2[1\cdot(x^2+y^2) - 2y\cdot y]}{(x^2+y^2)^2} = \frac{2[x^2+y^2-2y^2]}{(x^2+y^2)^2} = \frac{2(x^2-y^2)}{(x^2+y^2)^2} \quad \text{——— II}$$

Adding I and II

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2y^2-2x^2}{(x^2+y^2)^2} + \frac{2x^2-2y^2}{(x^2+y^2)^2} = \frac{2y^2-2x^2+2x^2-2y^2}{(x^2+y^2)^2}$$

$$= 0$$

proved

B.Sc. part-II (H), paper-III

Differential Calculus: Date- 20-04-2020

Q.2. If $u = t^{-\frac{3}{2}} e^{-\frac{r^2}{4Kt}}$, prove that

$$\frac{\partial u}{\partial t} = K\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\cdot\frac{\partial u}{\partial r}\right)$$

Soln: We have, $u = t^{-3/2} \cdot e^{-\frac{r^2}{4Kt}}$ —— (I)

Differentiating (I) partially w.r.t. t, keeping r constants, we get

$$\frac{\partial u}{\partial t} = -\frac{3}{2} t^{-\frac{5}{2}} \cdot e^{-\frac{r^2}{4Kt}} + t^{-\frac{3}{2}} e^{-\frac{r^2}{4Kt}} \left(-\frac{r^2}{4K}\right)\left(-\frac{1}{t^2}\right)$$

$$\Rightarrow \frac{\partial u}{\partial t} = -\frac{3}{2} t^{-\frac{5}{2}} e^{-\frac{r^2}{4Kt}} + \frac{r^2}{4K} t^{-\frac{7}{2}} e^{-\frac{r^2}{4Kt}} \quad \text{—— (II)}$$

Differentiating I partially w.r.t. to r (keeping t as constant), we get

$$\frac{\partial u}{\partial r} = t^{-\frac{3}{2}} e^{-\frac{r^2}{4Kt}} \left(-\frac{2r}{4Kt}\right)$$

$$\Rightarrow \frac{\partial u}{\partial r} = -\frac{t^{-\frac{5}{2}}}{2K} \left(r e^{-\frac{r^2}{4Kt}}\right) \quad \text{—— (III)}$$

Differentiating III w.r.t. to r (keeping t as constant), we get

$$\frac{\partial^2 u}{\partial r^2} = -\frac{t^{-\frac{5}{2}}}{2K}\left(e^{-\frac{r^2}{4Kt}} - \frac{2r^2}{4Kt}\right) \quad \text{—— (IV)}$$

$$\text{R.H.S} = K\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right)$$

$$= -\frac{1}{2} t^{-\frac{3}{2}} e^{-\frac{r^2}{4Kt}} + \frac{r^2}{4K} t^{-\frac{7}{2}} e^{-\frac{r^2}{4Kt}} - t^{-\frac{5}{2}} e^{-\frac{r^2}{4Kt}}$$

$$= -\frac{3}{2} t^{-\frac{5}{2}} e^{-\frac{r^2}{4Kt}} + \frac{r^2}{4K} t^{-\frac{7}{2}} e^{-\frac{r^2}{4Kt}}$$

$$= \frac{\partial u}{\partial t} = \text{L.H.S}$$

Proved.

B.sc - part - II (H), paper - III 20-04-2020

Differential Calculus

Q.3. If $y = f(x+ct) + \phi(x-ct)$, show that
$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

Soln: We have $y = f(x+ct) + \phi(x-ct)$ — (I)

Differentiating partially with respect to t, we get
$$\frac{\partial y}{\partial t} = f'(x+ct).c + \phi'(x-ct).(-c)$$
Again, differentiating this partially w.r.to. t, we get
$$\frac{\partial^2 y}{\partial t^2} = f''(x+ct)c^2 + \phi''(x-ct)(-c)^2$$
$$= c^2[f''(x+ct) + \phi''(x-ct)] \text{ — II}$$
Differentiating (I) partially with respect to x, we get
$$\frac{\partial y}{\partial x} = f'(x+ct).1 + \phi'(x-ct).1$$
Again, differentiating this partially with respect to x, we get
$$\frac{\partial^2 y}{\partial x^2} = f''(x+ct) + \phi''(x-ct) \text{ — III}$$
From II and III, we find that
$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$
Proved

prof. B.Sc-part-II (H), paper III 20-04-2020

Differential Calculus

Q.4. If ∇^2 stands for the operator

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$ prove that $\nabla^2 r = 0$.

where $\frac{1}{r^2} = x^2+y^2+z^2$

Soln. We have, $\frac{1}{r^2} = x^2+y^2+z^2$ —— (I)

Differentiating (I) partially w.r. to x (keeping y and z as constants), we get

$$-\frac{2}{r^3}\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{1}{r^2}\frac{\partial r}{\partial x} = -x \text{ —— (II)}$$

Again, differentiating (II) partially w.r. to x (keeping y and z as constants) we get

$$-\frac{3}{r^4}\left(\frac{\partial r}{\partial x}\right)^2 + \frac{1}{r^3}\cdot\frac{\partial^2 r}{\partial x^2} = -1$$

$$\Rightarrow -\frac{3}{r^4}(-xr^2)^2 + \frac{1}{r^3}\frac{\partial^2 r}{\partial x^2} = -1 \text{ [from II]}$$

$$\Rightarrow -3x^2r^2 + \frac{1}{r^3}\frac{\partial^2 r}{\partial x^2} = -1 \text{ —— (III)}$$

Similarly, $-3y^2r^2 + \frac{1}{r^3}\frac{\partial^2 r}{\partial y^2} = -1$ —— (IV)

and $-3z^2r^2 + \frac{1}{r^3}\frac{\partial^2 r}{\partial z^2} = -1$ —— (V)

Adding (III), (IV) and (V), we get

$$-3r^2(x^2+y^2+z^2) + \frac{1}{r^3}\left(\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}+\frac{\partial^2 r}{\partial z^2}\right) = -3$$

$$\Rightarrow -3\cancel{r^2}\cdot\frac{1}{\cancel{r^2}} + \frac{1}{r^3}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)r = -3 \text{ [from I]}$$

$$\Rightarrow \cancel{-3} + \frac{1}{r^3}\nabla^2 r = \cancel{-3}$$

Hence, $\nabla^2 r = 0$ Proved